Letters

Comments on and Extensions of "A Note on the Application of Edge-Elements for Modeling Three-Dimensional Inhomogeneously-Filled Cavities"

David B. Davidson

Abstract—**Some comments are made on the above paper[1] and a typographical error is corrected. Eigenvalue solutions for three-dimensional (3-D) cavities computed using an independently implemented finite-element code using edge-based elements are reported as verification of the tabulated formulas in this paper. The question of "trivial" eigenvalues is briefly addressed. The extension of the tabulated formulas to diagonally anisotropic media is presented; it is shown to be very straightforward. Such media currently have significant applications as artificial absorbers in finite-element meshes.**

Index Terms—**Artificial absorbers, diagonally anisotropic media, edge-based elements, finite-element method.**

I. Introduction

Differential-equation-based methods—especially the finite-difference time-domain (FDTD) method and finite-element method (FEM)—have achieved widespread acceptance in the microwave community for device simulation. The simplicity of the FDTD method lends itself to straightforward implementation; implementation of the FEM is somewhat more challenging. Explicit formulas for the $[S]$ and $[T]$ matrices for three-dimensional (3-D) edge-based elements are very useful in such an implementation, and are provided in this most useful paper,[1] which is surprisingly rarely cited by recent texts on the subject [1]–[3]. The formulas involve straightforward vector operations on edge and area vectors describing the edges and faces of the finite-element mesh.

The aim of this paper is threefold:

1) to correct a minor typographical error in the above paper,[1] and to provide verification of the expressions therein via an independent implementation;
2) to illustrate the accuracy of eigenvalue solution readily achievable and to comment on the "trivial" eigenvalues alluded to in the paper;
3) to give explicit formulas for the extensions to the case of anisotropic media (both electric and magnetic) with diagonal permittivity/permeability tensors.

Such materials are increasingly used to implement artificial (fictitious) absorbers, and are also encountered in microwave integrated circuits, and the required extensions to the tabulated formulas in the above paper[1] are very straightforward. Although the above paper[1] was published some time back, the material contained therein and the extensions to be outlined remain very topical; hence, the presentation of this paper.

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[1] J.-F. Lee and R. Mittra, *IEEE Trans. Microwave Theory Tech.*, vol. 40, no. 9, pp. 1767–1773, Sept. 1992.

TABLE I
Eight Lowest Nontrivial Eigenvalues (k_0, cm^{-1}), for a 1 cm × 0.5 cm × 0.75 cm Unloaded Rectangular Cavity

Mode No.	Analytical	Computed	|Error|
TE_{101}	5.236	5.209	0.52 %
TM_{110}	7.025	6.992	0.47 %
TE_{011}	7.531	7.505	0.35 %
TE_{201}	7.531	7.553	0.29 %
TM_{111}	8.179	8.031	1.81 %
TE_{111}	8.179	8.123	0.68 %
TM_{210}	8.886	8.753	1.50 %
TE_{102}	8.947	8.879	0.76 %

II. Corrections

The above paper[1] contains a minor typographical error. In the expression for T_{31} of (19), the entry should read

$$I_{00} - I_{01} - I_{03} + 2I_{13} \tag{1}$$

and not

$$I_{00} - I_{01} - I_{03} + 2I_{12}. \tag{2}$$

Min *et al.* correct this in [4].

Similar formulas are also given in Jin's text [1]; the expressions given in the above paper[1] are essentially the same as those in Jin [1, pp. 256–257]. Note that Jin's expressions for $[T_{15}]$, $[T_{25}]$, $[T_{35}]$, $[T_{45}]$, and $[T_{56}]$ (which, by symmetry, are the same as $[T_{51}], \cdots, [T_{65}]$) contain opposite signs to the above paper[1] due to the direction of Jin's edge five being opposite to that of Lee and Mittra. (From the errata, the first f_{34} in the last equation of [1, p. 256] should read f_{13}).

The term $[S_{6,3}]$ in (18) of the above paper[1] is somewhat unclear, presumably due to a printing error. It should read $\vec{t}_{0,1} \cdot \vec{t}_{1,2}$.

III. Numerical Determination of Eigenvalues with the FEM

This author has independently developed a program using the tabulated results in the above paper,[1] and has repeated various numerical experiments reported in the literature on the numerical determination of eigenvalues with the FEM. A brief summary will be presented. A study of an unloaded cavity reported by Jin [1, Sec. 8.4] has been performed; results are given in Table I for the eight lowest nontrivial eigenvalues for a 1 cm × 0.5 cm × 0.75 cm unloaded rectangular cavity, using edge-based tetrahedral elements. Mode numbering follows Jin [1, p. 260, Table 8.3]. The results shown were generated with a mesh with 351 degrees of freedom. (The nominal side length was 0.2 cm, corresponding to around ten edges per wavelength at the lowest resonant frequency of the cavity.) The agreement between analytical results (taken from [1, p. 260]) and computed results is quite satisfactory. Note that some of the eigenvalues are degenerate (e.g., TM_{111} and TE_{111}); the code does a reasonable approximation of these, but with nonidentical values.

Further confidence in the author's implementation, and, hence, the explicit formulas in the above paper[1] derives from results for the rectangular loaded cavity reported in the above paper[1]; results are given in Table II. The differences between the lowest nontrivial eigenvalues computed using these completely independently implemented codes (albeit using the same theory) is around 0.017%.

TABLE II
LOWEST NONTRIVIAL EIGENVALUE ka. EXACT/MEASURED DATA FROM THE ABOVE PAPER.[1] ERROR REFERS TO AUTHOR'S CODE WITH RESPECT TO EXACT/MEASURED DATA

Exact or measured	Lee and Mittra's FEM code	Present author's FEM code	|Error|
2.5829	2.59413	2.5937	0.418%

The meshes used in these studies were generated using Field Analysis Modeller (FAM), Version 4, a product of FEGS Ltd, Oakington, U.K. (This powerful commercial meshing program also has an integrated graphical post-processor). Packages such as these are crucial when developing a 3-D FEM program in a research environment, since developing robust 3-D meshers is a very challenging problem.

A point often not clearly made in the FEM literature on eigenvalue analysis is that a potentially large number of eigenvalues are computed by the eigenvalue routine, which must be rejected by the analyst. These are the approximations of the zero eigenvalues, as expected [5], [6]. Exactly how edge-based elements work in suppressing spurious modes has been the topic of much debate, but the emerging consensus (as expressed in [5] and [6]) is that they do a better job of approximating the zero-frequency eigenvalues, and not because the elements are divergence-free (which is, in any case, only true *within* the elements—and even this, only for lower order noncurvilinear elements).

Using edge-based elements, these approximations are—for problems such as those above—easily identified; for the problems reported above, the largest "zero-approximate" eigenvalue was typically two orders of magnitude smaller than the first desired eigenvalue. (For the unloaded cavity with the specific mesh used, the largest "trivial" eigenvalue—the 29th—was 0.027 98. This is a function of mesh size and type). This is implied in the literature by the term "lowest *nontrivial* eigenvalue." Peterson tabulates the number of zero eigenvalues for a particular analysis in [5, Table I]. It is actually possible to predict the number expected (see [7] in this regard), although this has not been attempted by this author.

IV. EXTENSION TO DIAGONALLY ANISOTROPIC MEDIA

The extension to anisotropic media in general is, unfortunately, somewhat tedious; this author has derived the case for a permittivity tensor with only diagonal, ϵ_{xy} and ϵ_{yx} entries, and the result is indeed lengthy. However, the extension to *diagonally* anisotropic media—a topic of current interest for fictitious absorber application, e.g., [8], as well as for uniaxial anisotropic materials, as encountered in integrated microwave circuits (see [1, p. 208] for some examples)—turns out to be very straightforward and will now be outlined. Several authors have published work based on this (e.g., [9]), and Min *et al.* [4] published formulas for anisotropic *dielectric* media, but other than this reference (which only handles dielectric anisotropy), explicit formulas are not available in the literature, to the best of this author's knowledge. (For anisotropic *waveguides*, the assumption of $e^{-\gamma z}$ propagation permits an efficient formulation involving two-dimensional discretization of the cross section [10].)

For an anisotropic material with *symmetric* permittivity and permeability tensors, real or complex (i.e., potentially lossy), the functional is (by analogy with [1, p. 221], note that the $1/\epsilon_r$ in [1, eq. (7.87)] is an error, present only in the first printing, or directly from [9, eq. (7)] with the last boundary integral term set to zero)

$$F(\vec{E}) = \int_\Omega [(\nabla \times \vec{E}) \cdot [\mu_r]^{-1} \cdot (\nabla \times \vec{E}) - k_0^2 \vec{E} \cdot [\epsilon_r] \cdot \vec{E}]\, d\Omega \quad (3)$$

$[\mu_r]$ and $[\epsilon_r]$ are the permittivity and permeability matrices/tensors/dyadics. Note that the $\cdot$ operation must now be interpreted as a tensor/dyadic operation. Jin includes a factor of 1/2 in the functional [1]; Sun and Balanis do not [9]. In this case, either form is acceptable, since the 1/2 factors out when the functional is minimized. (For permittivity and permeability tensors with Hermitian symmetry, the functional must be modified: see [1, p. 221]).

$$[S]_e = \frac{1}{9V}\begin{bmatrix} \vec{t}_{2,3}\cdot\vec{t'}_{2,3} & & & & & \\ -\vec{t}_{1,3}\cdot\vec{t'}_{2,3} & \vec{t}_{1,3}\cdot\vec{t'}_{1,3} & & & & \\ \vec{t}_{1,2}\cdot\vec{t'}_{2,3} & -\vec{t}_{1,2}\cdot\vec{t'}_{1,3} & \vec{t}_{1,2}\cdot\vec{t'}_{1,2} & & & \\ \vec{t}_{0,3}\cdot\vec{t'}_{2,3} & -\vec{t}_{0,3}\cdot\vec{t'}_{1,3} & \vec{t}_{0,3}\cdot\vec{t'}_{1,2} & \vec{t}_{0,3}\cdot\vec{t'}_{0,3} & & \\ -\vec{t}_{0,2}\cdot\vec{t'}_{2,3} & \vec{t}_{0,2}\cdot\vec{t'}_{1,3} & -\vec{t}_{0,2}\cdot\vec{t'}_{1,2} & -\vec{t}_{0,2}\cdot\vec{t'}_{0,3} & \vec{t}_{0,2}\cdot\vec{t'}_{0,2} & \\ \vec{t}_{0,1}\cdot\vec{t'}_{2,3} & -\vec{t}_{0,1}\cdot\vec{t'}_{1,3} & \vec{t}_{0,1}\cdot\vec{t'}_{1,2} & \vec{t}_{0,1}\cdot\vec{t'}_{0,3} & -\vec{t}_{0,1}\cdot\vec{t'}_{0,2} & \vec{t}_{0,1}\cdot\vec{t'}_{0,1} \end{bmatrix} \quad (7)$$

$$[T]_e = \frac{1}{180V}\begin{bmatrix} \begin{array}{c}2(I'_{00}\\ -I'_{01}+I'_{11})\end{array} & & & & & \\ \begin{array}{c}I'_{00}-I'_{01}\\ -I'_{02}+2I'_{12}\end{array} & \begin{array}{c}2(I'_{00}-\\ I'_{02}+I'_{22})\end{array} & & & & \\ \begin{array}{c}I'_{00}-I'_{01}\\ -I'_{03}+2I'_{13}\end{array} & \begin{array}{c}I'_{00}-I'_{02}\\ -I'_{03}+2I'_{23}\end{array} & \begin{array}{c}2(I'_{00}-\\ I'_{03}+I'_{33})\end{array} & & & \\ \begin{array}{c}I'_{01}-I'_{11}\\ -2I'_{02}+I'_{12}\end{array} & \begin{array}{c}2I'_{01}-I'_{12}\\ -I'_{02}+I'_{22}\end{array} & \begin{array}{c}I'_{01}-I'_{13}\\ -I'_{02}+I'_{23}\end{array} & \begin{array}{c}2(I'_{11}-\\ I'_{12}+I'_{22})\end{array} & & \\ \begin{array}{c}I'_{01}-I'_{11}\\ -2I'_{03}+I'_{13}\end{array} & \begin{array}{c}I'_{01}-I'_{12}\\ -I'_{03}+I'_{23}\end{array} & \begin{array}{c}2I'_{01}-I'_{13}\\ -I'_{03}+I'_{33}\end{array} & \begin{array}{c}I'_{11}-I'_{12}\\ -I'_{13}+2I'_{23}\end{array} & \begin{array}{c}2(I'_{11}-\\ I'_{13}+I'_{33})\end{array} & \\ \begin{array}{c}I'_{02}-I'_{12}\\ -I'_{03}+I'_{13}\end{array} & \begin{array}{c}I'_{02}-I'_{22}\\ -2I'_{03}+I'_{23}\end{array} & \begin{array}{c}2I'_{02}-I'_{23}\\ -I'_{03}+I'_{33}\end{array} & \begin{array}{c}I'_{12}-I'_{22}\\ -2I'_{13}+I'_{23}\end{array} & \begin{array}{c}2I'_{12}-I'_{23}\\ -I'_{13}+I'_{33}\end{array} & \begin{array}{c}2(I'_{22}-\\ I'_{23}+I'_{33})\end{array} \end{bmatrix} \quad (8)$$

With diagonal permittivity and permeability tensors given by

$$[\epsilon_r] = \begin{bmatrix} \epsilon_{r_{xx}} & 0 & 0 \\ 0 & \epsilon_{r_{yy}} & 0 \\ 0 & 0 & \epsilon_{r_{zz}} \end{bmatrix} \tag{4}$$

$$[\mu_r] = \begin{bmatrix} \mu_{r_{xx}} & 0 & 0 \\ 0 & \mu_{r_{yy}} & 0 \\ 0 & 0 & \mu_{r_{zz}} \end{bmatrix} \tag{5}$$

all that is required are the following modifications. In the evaluation of the $[S]$ matrix elements, all the dot products of the form $\vec{t}_{i,j} \cdot \vec{t}_{k,l}$ must be replaced by $\vec{t}_{i,j} \cdot \vec{t'}_{k,l}$ with

$$\vec{t'}_{k,l} = \frac{1}{\mu_{r_{xx}}} t_{k,l_x} + \frac{1}{\mu_{r_{yy}}} t_{k,l_y} + \frac{1}{\mu_{r_{zz}}} t_{k,l_z} = [\mu_r]^{-1} \cdot \vec{t}_{k,1} \tag{6}$$

where $\cdot$ is now the conventional matrix–vector inner product. Note also that the $1/\mu_r$ term in (18) of the above paper [1] must be removed since it is now effectively included in the dot product. The matrix retains its symmetry. The lower half and diagonal of the $[S]$ matrix is shown in (7), at the bottom of the previous page.

For the $[T]$ matrix elements, Min *et al.* [4] provide the required expressions, but since this conference publication may not be readily accessible, the extension will be given here. (It was derived independently by this author.) Terms of the form $I_{ij} = \vec{A}_i \cdot \vec{A}_j$ must be replaced by $I'_{ij} = \vec{A}_i \cdot \vec{A}'_j$ with $\vec{A}'_j = [\epsilon_{r_{xx}} A_{j_x} + \epsilon_{r_{yy}} A_{j_y} + \epsilon_{r_{zz}} A_{j_z}] = [\epsilon_r] \cdot \vec{A}_j$. The ϵ_r term in (19) of the above paper[1] must also be removed since it is now included in I'_{ij}. The lower half and diagonal of the $[T]$ matrix is shown in (8), at the bottom of the previous page.

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Comments on "Transitional Combline/Evanescent-Mode Microwave Filters"

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In the above paper,[1] Levy *et al.* refer to the phenomenon of combline-filter bandwidth expansion (i.e., practical bandwidth versus theoretical TEM-analyzed bandwidth). Levy *et al.* explain that this phenomenon is mainly caused due to evanescent waveguide modes propagating through the structure, affecting the overall coupling coefficients and bandwidth. This explanation, known for many years, is only one among other explanations such as coupling between nonadjacent resonators, also known for many years. These explanations and derived equivalent models are not fully compliant with practical results and may be applicable only in limited frequencies and structural dimension.

In paragraph five of the above-paper,[1] Levy *et al.* claim to have investigated and disprove an explanation suggested by us which was recently published [1]. This explanation is based on deviation from quasi-static two-dimensional cross-sectional TEM-derived coupling coefficients, mainly caused due to the proximity of a ground plane to the open ends of the resonator array, significantly affecting the overall bandwidth. The effect of this ground plane, usually used to carry tuning elements, is not fully represented in traditional equivalent models and design formulas for combline-filter design and analysis.

However, Levy *et al.* investigated a structure with a large iris between the resonators, which is significantly different than the classic structure we have investigated. Therefore, the "disproval" of our explanation by Levy *et al.* has no practical validation.

Moreover, it is expected that evanescent waveguide modes should cause similar effects in interdigital filters, yet these filters' performance comply with their TEM analysis, a fact Levy *et al.* admit to be unable to explain in paragraph six of the above paper.[1] According to our explanation, this fact is obvious since in interdigital filters the resonator open ends hardly participate in the overall coupling. In addition, Levy *et al.* do not explain the dependence of that phenomenon on the spacing between resonators in paragraph four of the above paper,[1] while our explanation is consistent with this

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